

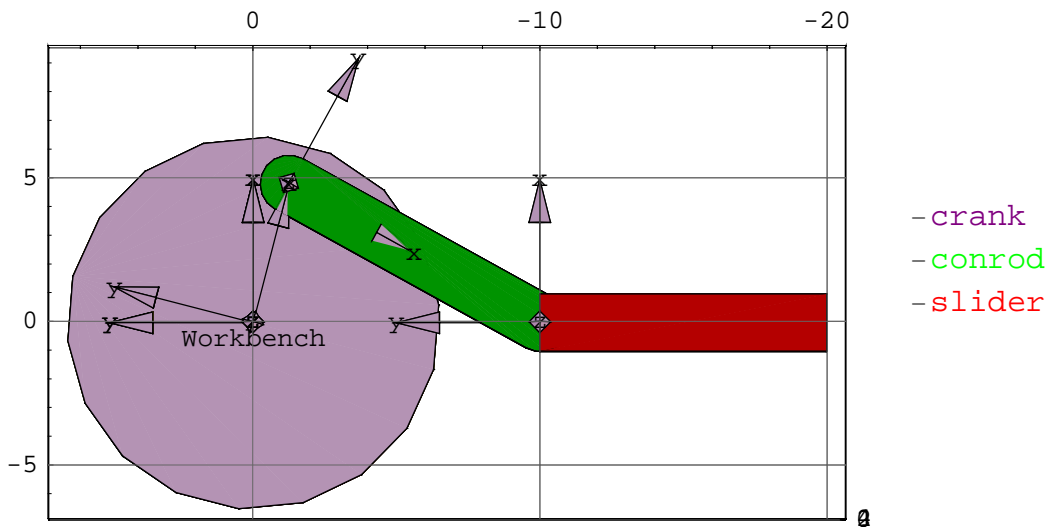
Crank-Slider mechanism

Singularity Analysis

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In this notebook the singularity analysis of the crank slider mechanism is considered. The mathematical model of this one looped mechanism is built in LinkageDesigner using the slider link as the driving part. It is well known, that if the slider is moved back and forth between its two extremity, the crank rotates continuously. This continuous rotation is attributed partly to the inertia of the crank, which is not taken into account in the kinematic model of the mechanism. In case of the "virtual crank-slider mechanism", therefore one has to substitute the physics with some methodology. This continuous rotation of the crank is quite challenging problem. The problem is laying in the solution of the loop closing equations (see LinkageDesigner 0.1.2). The loop closing equation in our case has two solution for one given position of the slider. This two solution branch collapse into one solution in the singular position of the mechanism. The challenge is the following: in order to rotate continuously the crank, one has to ensure that the mechanism switches solution branches as he touches its singular position.



Crank-Slider mechanism

Build the mechanism

Load the package

```
In[1]:= << LinkageDesigner`
```

Create the LinkageData object with two geometrical parameters l1 (as the length of crank) and l2 (as the length of conrod)

```
In[2]:= cr = CreateLinkage["cr", SimpleParameters -> {l1 -> 5, l2 -> 10}]
```

```
Out[2]:= -LinkageData, 6-
```

Define the translational joint between Workbench and slider

```
In[3]:= DefineKinematicPairTo[cr, "Translational", {posSlider},
  {"Workbench", MakeHomogenousMatrix[RotationMatrix[{1, 0, 0},  $\pi/2$ ]]},
  {"slider", MakeHomogenousMatrix[RotationMatrix[{1, 0, 0},  $\pi/2$ ]]}]
```

```
Out[3]:= -LinkageData, 6-
```

Define the rotational joint between slider and conrod

```
In[4]:= DefineKinematicPairTo[cr, "Rotational", { $\theta_2$ },
  {"slider", MakeHomogenousMatrix[{0, 0, 0}]},
  {"conrod", MakeHomogenousMatrix[{12, 0, 0}]}]
```

```
Out[4]:= -LinkageData, 6-
```

Define the rotational joint between slider and crank

```
In[5]:= DefineKinematicPairTo[cr, "Rotational", { $\theta_1$ },
  {"crank", MakeHomogenousMatrix[{0, 0, 0}]},
  {"Workbench", MakeHomogenousMatrix[{0, 0, 0}]}]
```

```
Out[5]:= -LinkageData, 6-
```

Move the mechanism out from singular position in order to define the loop closing equations

```
In[6]:= SetDrivingVariablesTo[cr, { $\theta_2$  -> 30°,  $\theta_1$  -> 40°},
  MaxIterations -> 150, AccuracyGoal -> 8]
```

```
Out[6]:= -LinkageData, 6-
```

Close the kinematic graph with a rotational joint between crank and conrod

```

In[7]:= DefineKinematicPairTo[cr, "Rotational", {q3},
  {"conrod", MakeHomogenousMatrix[{0, 0, 0}]},
  {"crank", MakeHomogenousMatrix[{11, 0, 0}]}, Verbose → True]
This is a loop-closing kinematic pair.
Placing links into constrained position is finished in 0.02[sec]
Translational blocked directions: {{0., 0., 1.}}
Rotational blocked directions: {{0., -1., 1.11022×10-16}, {1., 0., 0.}}
Candidate loop variables: {posSlider,  $\theta_2$ ,  $\theta_1$ }
Non redundant constraint equations: 2
Selected loop variables: { $\theta_1$ ,  $\theta_2$ }
The updated list of driving variables: {posSlider → -5.02457}

```

```
Out[7]= -LinkageData, 7-
```

Check the generated loop closing equations (for enhance the visibility substitute $\{-1.^{16} \rightarrow -1\}$)

```

In[8]:= Simplify[cr[["$DerivedParametersB"]] /. {-1.^16. → -1} // TableForm
Out[8]//TableForm=
  cr@RotationalJoint-4       $\theta_2 \rightarrow 14.0673$        $11 + 12 \text{Cos}[\theta_1 + \theta_2] = \text{posSlider} \text{Sin}[\theta_1]$ 
                           $\theta_1 \rightarrow 14.2772$        $\text{posSlider} \text{Cos}[\theta_1] + 12 \text{Sin}[\theta_1 + \theta_2] = 0$ 

```

Define geometry and attache to the links

```

In[9]:= cr[["$LinkGeometry", "crank"]] =
  PlaceShape[Graphics3D[LinkShape[0, 11 * 1.3, 11 * 1.3, 0.1]], {0, 0, -1}];
cr[["$LinkGeometry", "conrod"]] =
  Graphics3D[{SurfaceColor[Green], LinkShape[12, 1, 1, 0.1]};
cr[["$LinkGeometry", "slider"]] =
  Graphics3D[{SurfaceColor[Red], Cuboid[{-1, 0, -1}, {1, -10, 1}]}];

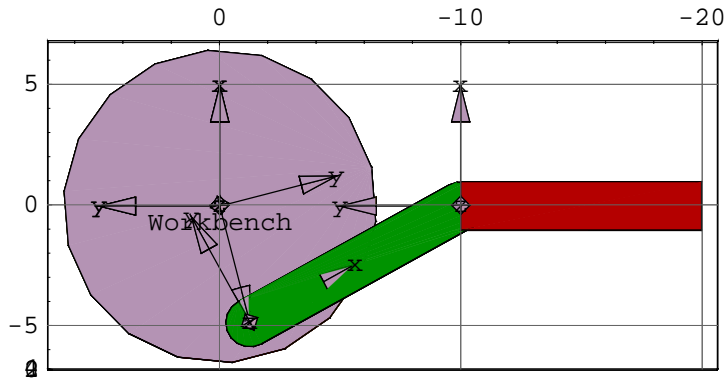
```

In order to visualize the mechanism move the slider into the position 10. This can be done using the SetDrivingVariablesTo function. Since the mechanism contains implicitly derived parameters, the SetDrivingVariables function takes care to solve the constraint equations corresponding to the derived parameters (in our case to θ_1 and θ_2).

The constraint equations usually has more than one solution branches, of which the root finding procedure will find one. To control the result of the root finding the initial guess of the parameters can be changed. The Initial guess are the starting point for the root finding procedure, that influences the solution branch will be hit.

Drive and display the mechanism using $\theta_1 \rightarrow -2$ as the initial guess for the root finding

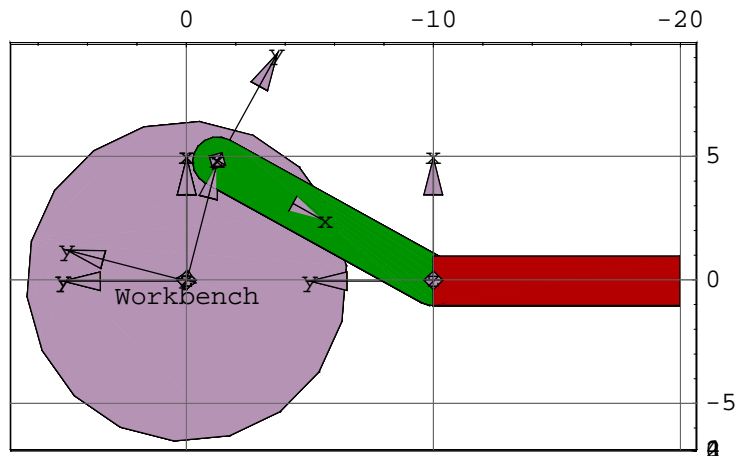
```
In[12]:= Show[Linkage3D[
  SetDrivingVariablesTo[cr, {posSlider → 10}, ParameterInitialGuess → {θ1 → -2}],
  LinkMarkers -> All, MarkerSize → 5],
  PlotRange -> All, ViewPoint → {-1, 0, 100}, Axes → True, FaceGrids → {{0, 0, 1}}]
```



Out[12]= - Graphics3D -

Drive and display the mechanism using $\theta_1 \rightarrow 1$ as the initial guess for the root finding

```
In[13]:= Show[Linkage3D[
  SetDrivingVariablesTo[cr, {posSlider → 10}, ParameterInitialGuess → {θ1 → 1}],
  LinkMarkers -> All, MarkerSize → 5],
  PlotRange -> All, ViewPoint → {-1, 0, 100}, Axes → True, FaceGrids → {{0, 0, 1}}]
```



Out[13]= - Graphics3D -

Singularity Analysis

Here I will not go into detail of the mathematics of the singularity analysis. One can find it in standard text books of analysis. It is enough to say that the singular points of the constraint equations can be calculated if the Jacobian of the constraint equations are calculated. The singular points are determined if this determinant of the Jacobian is equal to 0.

■ Get the left side of the constraint equation (the right side is 0)

```
In[14]:= eq = cr [["$DerivedParametersB"]] [[1, 3, All, 1]] /. {-1.16. -> -1}
```

```
Out[14]= {-11 - 12 Cos[θ1] Cos[θ2] + posSlider Sin[θ1] + 12 Sin[θ1] Sin[θ2],
          -posSlider Cos[θ1] - 12 Cos[θ2] Sin[θ1] - 12 Cos[θ1] Sin[θ2]}
```

■ Calculate the Jacobian and the determinant of the Jacobian

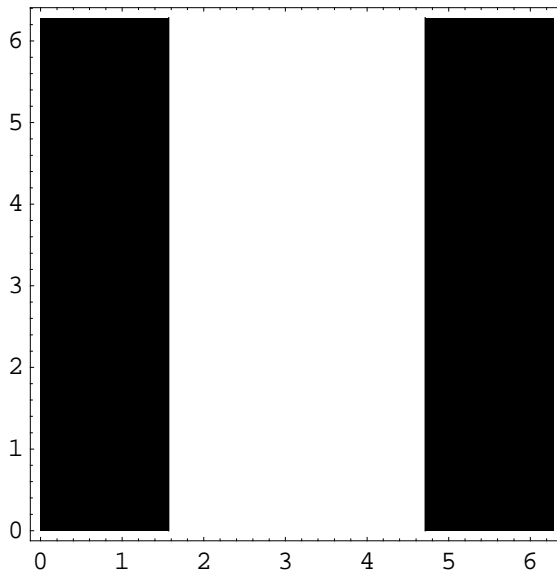
```
In[15]:= Jmx = Outer[D, eq, {θ1, θ2}]; MatrixForm[Simplify[Jmx]]
DetJmx = Chop[TrigFactor[Det[Jmx]]]
```

```
Out[15]//MatrixForm=
( posSlider Cos[θ1] + 12 Sin[θ1 + θ2]    12 Sin[θ1 + θ2]
  -12 Cos[θ1 + θ2] + posSlider Sin[θ1]  -12 Cos[θ1 + θ2] )
```

```
Out[16]= -12 posSlider Cos[θ2]
```

■ Plot the determinant of the Jacobian in the {θ2,θ3} plane. Display only the contour Det(Jacobian) =0 contour

```
In[17]:= dfunc = DetJmx /. GetLinkageRules[cr, IncludeDerivedParametersB -> False];
gr = ContourPlot[dfunc, {θ2, 0, 2 π}, {θ3, 0, 2 π}, PlotPoints -> 100, Contours -> {0}]
```



```
Out[18]= - ContourGraphics -
```

It is easy to see that the singular position of the mechanism is $\theta_2 = \pi/2 + k\pi$. The slider's position in the singular posture is 5 and 15. The challenge is to rotate the crank one whole circle while the slider is moved from 15 to 5 and back to 15. There are two possible solutions for this problem

- Using numerical root finding and changing the initial values of the solver
- Solving analytically the loop closing equations in closed form and switching between solution branches

■ Numerical solution

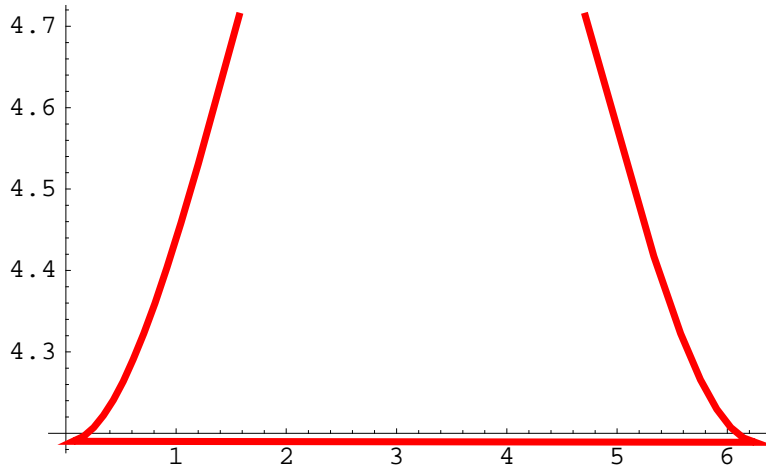
Move the slider from position 15 to 5 by using the first solution branch, and calculate the substitution values of all parameters

```
In[19]:= SetDrivingVariablesTo[cr,
  {posSlider -> 14.9999}, ParameterInitialGuess -> {θ1 -> 1}];
ls1 = GetLinkageRules[cr, {{posSlider -> 15}, {posSlider -> 5}}, Resolution -> 20,
  MaxIterations -> 150, AccuracyGoal -> 8, SubstituteParameters -> True]
```

```
Out[20]= {{posSlider -> 15, l1 -> 5, l2 -> 10, θ2 -> -58.1195, θ1 -> 1.5708},
  {posSlider ->  $\frac{29}{2}$ , l1 -> 5, l2 -> 10, θ2 -> -58.3007, θ1 -> 1.20199},
  {posSlider -> 14, l1 -> 5, l2 -> 10, θ2 -> -58.3737, θ1 -> 1.04373},
  {posSlider ->  $\frac{27}{2}$ , l1 -> 5, l2 -> 10, θ2 -> -58.428, θ1 -> 0.918092},
  {posSlider -> 13, l1 -> 5, l2 -> 10, θ2 -> -58.4721, θ1 -> 0.808246},
  {posSlider ->  $\frac{25}{2}$ , l1 -> 5, l2 -> 10, θ2 -> -58.5092, θ1 -> 0.707584},
  {posSlider -> 12, l1 -> 5, l2 -> 10, θ2 -> -58.5409, θ1 -> 0.612604},
  {posSlider ->  $\frac{23}{2}$ , l1 -> 5, l2 -> 10, θ2 -> -58.568, θ1 -> 0.52109},
  {posSlider -> 11, l1 -> 5, l2 -> 10, θ2 -> -58.5909, θ1 -> 0.431443},
  {posSlider ->  $\frac{21}{2}$ , l1 -> 5, l2 -> 10, θ2 -> -58.6099, θ1 -> 0.342363},
  {posSlider -> 10, l1 -> 5, l2 -> 10, θ2 -> -58.6248, θ1 -> 0.25268},
  {posSlider ->  $\frac{19}{2}$ , l1 -> 5, l2 -> 10, θ2 -> -58.6356, θ1 -> 0.161224},
  {posSlider -> 9, l1 -> 5, l2 -> 10, θ2 -> -58.6418, θ1 -> 0.0667161},
  {posSlider ->  $\frac{17}{2}$ , l1 -> 5, l2 -> 10, θ2 -> -58.6428, θ1 -> -0.0323586},
  {posSlider -> 8, l1 -> 5, l2 -> 10, θ2 -> -58.6376, θ1 -> -0.137937},
  {posSlider ->  $\frac{15}{2}$ , l1 -> 5, l2 -> 10, θ2 -> -58.6248, θ1 -> -0.25268},
  {posSlider -> 7, l1 -> 5, l2 -> 10, θ2 -> -58.6022, θ1 -> -0.380547},
  {posSlider ->  $\frac{13}{2}$ , l1 -> 5, l2 -> 10, θ2 -> -58.5661, θ1 -> -0.528046},
  {posSlider -> 6, l1 -> 5, l2 -> 10, θ2 -> -58.5092, θ1 -> -0.707584},
  {posSlider ->  $\frac{11}{2}$ , l1 -> 5, l2 -> 10, θ2 -> -58.4144, θ1 -> -0.95038},
  {posSlider -> 5, l1 -> 5, l2 -> 10, θ2 -> -58.1195, θ1 -> -1.5708}}
```

■ Display the trajectory if the $\{\theta_1, \theta_2\}$ coordinate frame

```
In[21]:= gr1 = ListPlot[{Mod[ $\theta_1$ ,  $2\pi$ ], Mod[ $\theta_2$ ,  $2\pi$ ]} /. ls1,
  PlotJoined -> True, PlotStyle -> {Red, Thickness[0.01]}]
```



```
Out[21]= - Graphics -
```

```
In[22]:= SetDrivingVariablesTo[cr, {posSlider -> 5.0001},
  MaxIterations -> 150, AccuracyGoal -> 8, ParameterInitialGuess -> { $\theta_1$  -> -1}]
```

```
Out[22]= -LinkageData, 7-
```

Move the slider from position $5 + dx$ by using the second solution branch, and calculate the substitution values of all parameters, then the slider moves back from position 5 to 15

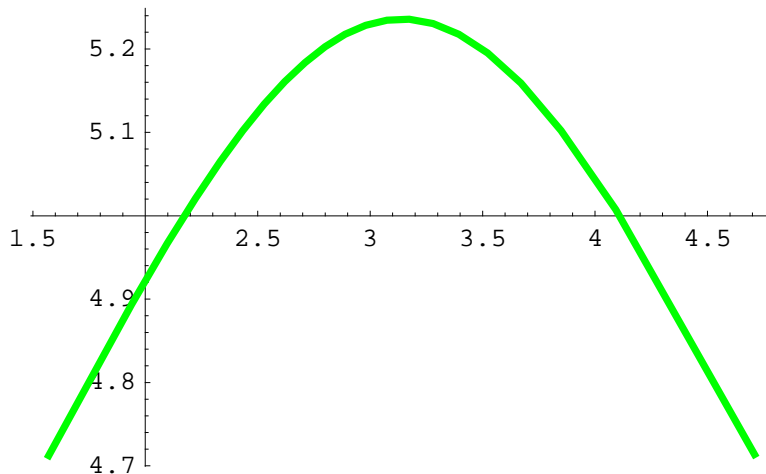
```
In[23]:= SetDrivingVariablesTo[cr, {posSlider -> 5.0001}, MaxIterations -> 150,
  AccuracyGoal -> 8, ParameterInitialGuess -> {θ1 -> -1}]
ls2 = GetLinkageRules[cr, {{posSlider -> 5.00001}, {posSlider -> 15}},
  Resolution -> 20, SubstituteParameters -> True]
```

```
Out[23]= -LinkageData, 7-
```

```
Out[24]= {{posSlider -> 5.00001, l1 -> 5, l2 -> 10, θ2 -> -58.118, θ1 -> -1.57362},
  {posSlider -> 5.50001, l1 -> 5, l2 -> 10, θ2 -> -57.8245, θ1 -> -2.19122},
  {posSlider -> 6.00001, l1 -> 5, l2 -> 10, θ2 -> -57.7297, θ1 -> -2.43401},
  {posSlider -> 6.50001, l1 -> 5, l2 -> 10, θ2 -> -57.6729, θ1 -> -2.61355},
  {posSlider -> 7.00001, l1 -> 5, l2 -> 10, θ2 -> -57.6367, θ1 -> -2.76105},
  {posSlider -> 7.50001, l1 -> 5, l2 -> 10, θ2 -> -57.6141, θ1 -> -2.88891},
  {posSlider -> 8.00001, l1 -> 5, l2 -> 10, θ2 -> -57.6013, θ1 -> -3.00366},
  {posSlider -> 8.50001, l1 -> 5, l2 -> 10, θ2 -> -57.5962, θ1 -> -3.10924},
  {posSlider -> 9.00001, l1 -> 5, l2 -> 10, θ2 -> -57.5971, θ1 -> -3.20831},
  {posSlider -> 9.50001, l1 -> 5, l2 -> 10, θ2 -> -57.6033, θ1 -> -3.30282},
  {posSlider -> 10., l1 -> 5, l2 -> 10, θ2 -> -57.6141, θ1 -> -3.39427},
  {posSlider -> 10.5, l1 -> 5, l2 -> 10, θ2 -> -57.6291, θ1 -> -3.48396},
  {posSlider -> 11., l1 -> 5, l2 -> 10, θ2 -> -57.648, θ1 -> -3.57304},
  {posSlider -> 11.5, l1 -> 5, l2 -> 10, θ2 -> -57.6709, θ1 -> -3.66268},
  {posSlider -> 12., l1 -> 5, l2 -> 10, θ2 -> -57.698, θ1 -> -3.7542},
  {posSlider -> 12.5, l1 -> 5, l2 -> 10, θ2 -> -57.7297, θ1 -> -3.84918},
  {posSlider -> 13., l1 -> 5, l2 -> 10, θ2 -> -57.7668, θ1 -> -3.94984},
  {posSlider -> 13.5, l1 -> 5, l2 -> 10, θ2 -> -57.8109, θ1 -> -4.05969},
  {posSlider -> 14., l1 -> 5, l2 -> 10, θ2 -> -57.8652, θ1 -> -4.18532},
  {posSlider -> 14.5, l1 -> 5, l2 -> 10, θ2 -> -57.9382, θ1 -> -4.34358},
  {posSlider -> 15., l1 -> 5, l2 -> 10, θ2 -> -58.1195, θ1 -> -4.71239}}
```

Display the trajectory if the $\{\theta_1, \theta_2\}$ coordinate frame

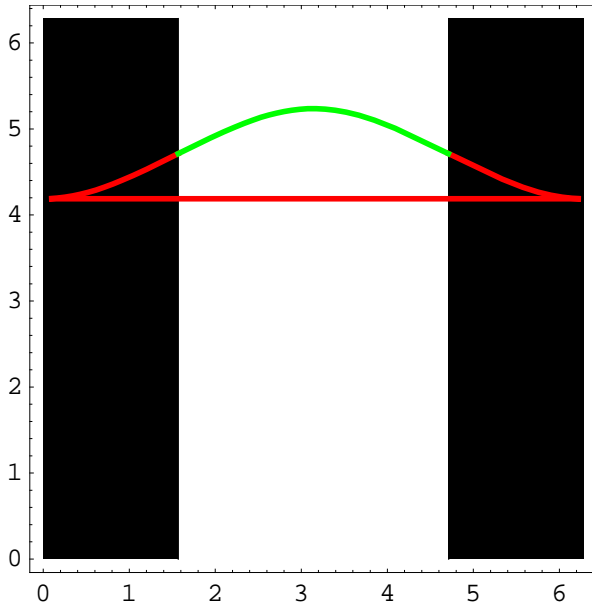
```
In[25]:= gr2 = ListPlot[{Mod[θ1, 2 π], Mod[θ2, 2 π]} /. ls2,
  PlotJoined -> True, PlotStyle -> {Green, Thickness[0.01]]}
```



```
Out[25]= - Graphics -
```

Display the trajectory if the $\{\theta_1, \theta_2\}$ coordinate frame as the slider moves from position 15 to 5 and than back to position 5

```
In[26]:= Show[{gr, gr1, gr2}]
```



```
Out[26]= - Graphics -
```

Animate the linkage to verify the result

```
In[27]:= ls = AnimateLinkage[cr, Join[Drop[ls1, 1], Drop[ls2, 1]],
  Resolution -> None, LinkMarkers -> All, MarkerSize -> 4,
  ViewPoint -> {-1, 0, 100}, Axes -> True, FaceGrids -> {{0, 0, 1}}];
```

■ Analytical solution

Use the unit circle substitution convert the trigonometrical polinom into a polinom

```
In[29]:= sub = {Cos[θ1] -> c1, Cos[θ2] -> c2, Sin[θ1] -> s1, Sin[θ2] -> s2};
  invsub = Reverse /@ sub
```

```
Out[30]= {c1 -> Cos[θ1], c2 -> Cos[θ2], s1 -> Sin[θ1], s2 -> Sin[θ2]}
```

```
In[31]:= sys = Join[TrigExpand[eq] /. sub, {c12 + s12, c22 + s22}] == {0, 0, 1, 1}
```

```
Out[31]= {-l1 - c1 c2 l2 + posSlider s1 + l2 s1 s2,
  -c1 posSlider - c2 l2 s1 - c1 l2 s2, c12 + s12, c22 + s22} == {0, 0, 1, 1}
```

Solve the resulted set of polinom equations with the Solve function

```
In[32]:= sol = Solve[sys, {c1, s1, c2, s2}]
```

$$\text{Out[32]} = \left\{ \left\{ c2 \rightarrow -\frac{11 \sqrt{1 - \frac{(11^2 - 12^2 + \text{posSlider}^2)^2}{4 11^2 \text{posSlider}^2}}}{12}, s2 \rightarrow \frac{11^2 - 12^2 - \text{posSlider}^2}{2 12 \text{posSlider}}, \right. \right. \\ \left. \left. c1 \rightarrow \sqrt{1 - \frac{(11^2 - 12^2 + \text{posSlider}^2)^2}{4 11^2 \text{posSlider}^2}}, s1 \rightarrow \frac{11^2 - 12^2 + \text{posSlider}^2}{2 11 \text{posSlider}} \right\}, \right. \\ \left. \left\{ c2 \rightarrow \frac{11 \sqrt{1 - \frac{(11^2 - 12^2 + \text{posSlider}^2)^2}{4 11^2 \text{posSlider}^2}}}{12}, s2 \rightarrow \frac{11^2 - 12^2 - \text{posSlider}^2}{2 12 \text{posSlider}}, \right. \right. \\ \left. \left. c1 \rightarrow -\sqrt{1 - \frac{(11^2 - 12^2 + \text{posSlider}^2)^2}{4 11^2 \text{posSlider}^2}}, s1 \rightarrow \frac{11^2 - 12^2 + \text{posSlider}^2}{2 11 \text{posSlider}} \right\} \right\}$$

Substitute back the solution into the derived parameters

```
In[33]:= repl = {θ1 → ArcTan[c1, s1], θ2 → ArcTan[c2, s2]} /. sol
```

$$\text{Out[33]} = \left\{ \left\{ \theta1 \rightarrow \text{ArcTan} \left[\sqrt{1 - \frac{(11^2 - 12^2 + \text{posSlider}^2)^2}{4 11^2 \text{posSlider}^2}}, \frac{11^2 - 12^2 + \text{posSlider}^2}{2 11 \text{posSlider}} \right], \right. \right. \\ \left. \left. \theta2 \rightarrow \text{ArcTan} \left[-\frac{11 \sqrt{1 - \frac{(11^2 - 12^2 + \text{posSlider}^2)^2}{4 11^2 \text{posSlider}^2}}}{12}, \frac{11^2 - 12^2 - \text{posSlider}^2}{2 12 \text{posSlider}} \right] \right\}, \right. \\ \left. \left\{ \theta1 \rightarrow \text{ArcTan} \left[-\sqrt{1 - \frac{(11^2 - 12^2 + \text{posSlider}^2)^2}{4 11^2 \text{posSlider}^2}}, \frac{11^2 - 12^2 + \text{posSlider}^2}{2 11 \text{posSlider}} \right], \right. \right. \\ \left. \left. \theta2 \rightarrow \text{ArcTan} \left[\frac{11 \sqrt{1 - \frac{(11^2 - 12^2 + \text{posSlider}^2)^2}{4 11^2 \text{posSlider}^2}}}{12}, \frac{11^2 - 12^2 - \text{posSlider}^2}{2 12 \text{posSlider}} \right] \right\} \right\}$$

Since $\{\theta_1, \theta_2\}$ are calculated as an explicit function of the DrivingVariables and the SimpleParameters the \$DerivedParametersB record can be replaced with a corresponding \$DerivedParametersA record. This way one can define two resolved linkage, that has its loop closure equations solved in closed form.

Append the first solution branch of the solution to the \$DerivedParametersA record and remove \$DerivedParametersB record. Then set the resulted linkage to cr1

```
In[34]:= cr1 = Append[cr, {"$DerivedParametersA", repl[[1]]}]
cr1 = Delete[cr1, "$DerivedParametersB"]
```

```
Out[34]= -LinkageData, 8-
```

```
Out[35]= -LinkageData, 7-
```

Append the second solution branch of the solution to the \$DerivedParametersA record and remove \$DerivedParametersB record. Then set the resulted linkage to cr2

```
In[36]:= cr2 = Append[cr, {"$DerivedParametersA", repl[[2]]}]
cr2 = Delete[cr2, "$DerivedParametersB"]
```

```
Out[36]= -LinkageData, 8-
```

```
Out[37]= -LinkageData, 7-
```

Move the slider from position 15 to 5 by using the first solution branch, and calculate the substitution values of all parameters

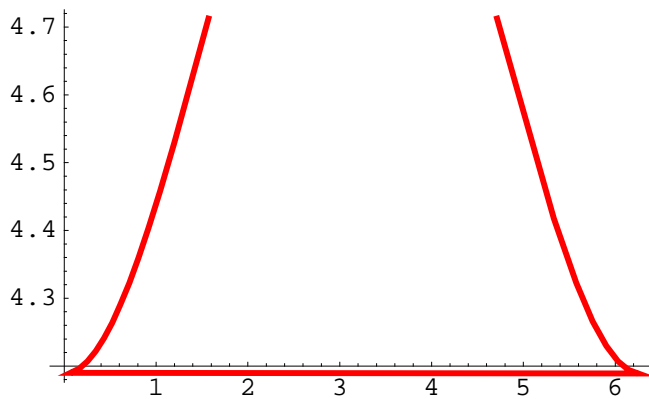
```
In[38]:= ls1 = GetLinkageRules[cr1, {{posSlider -> 15}, {posSlider -> 5}}, Resolution -> 20,
MaxIterations -> 150, AccuracyGoal -> 8, SubstituteParameters -> True]
```

```
Out[38]= {{posSlider -> 15, l1 -> 5, l2 -> 10,  $\theta_1 \rightarrow \frac{\pi}{2}$ ,  $\theta_2 \rightarrow -\frac{\pi}{2}$ }, {posSlider ->  $\frac{29}{2}$ ,
l1 -> 5, l2 -> 10,  $\theta_1 \rightarrow \text{ArcTan}\left[\frac{541}{\sqrt{43719}}\right]$ ,  $\theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{1141}{\sqrt{43719}}\right]$ },
{posSlider -> 14, l1 -> 5, l2 -> 10,  $\theta_1 \rightarrow \text{ArcTan}\left[\frac{121}{3\sqrt{551}}\right]$ ,  $\theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{271}{3\sqrt{551}}\right]$ },
{posSlider ->  $\frac{27}{2}$ , l1 -> 5, l2 -> 10,
 $\theta_1 \rightarrow \text{ArcTan}\left[\frac{143}{\sqrt{11951}}\right]$ ,  $\theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{343}{\sqrt{11951}}\right]$ },
{posSlider -> 13, l1 -> 5, l2 -> 10,  $\theta_1 \rightarrow \text{ArcTan}\left[\frac{47}{12\sqrt{14}}\right]$ ,  $\theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{61}{6\sqrt{14}}\right]$ },
{posSlider ->  $\frac{25}{2}$ , l1 -> 5, l2 -> 10,  $\theta_1 \rightarrow \text{ArcTan}\left[\frac{13}{\sqrt{231}}\right]$ ,  $\theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{37}{\sqrt{231}}\right]$ },
{posSlider -> 12, l1 -> 5, l2 -> 10,  $\theta_1 \rightarrow \text{ArcTan}\left[\frac{23}{3\sqrt{119}}\right]$ ,  $\theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{73}{3\sqrt{119}}\right]$ },
{posSlider ->  $\frac{23}{2}$ , l1 -> 5, l2 -> 10,
 $\theta_1 \rightarrow \text{ArcTan}\left[\frac{229}{\sqrt{159159}}\right]$ ,  $\theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{829}{\sqrt{159159}}\right]$ },
{posSlider -> 11, l1 -> 5, l2 -> 10,  $\theta_1 \rightarrow \text{ArcTan}\left[\frac{23}{8\sqrt{39}}\right]$ ,  $\theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{49}{4\sqrt{39}}\right]$ },
{posSlider ->  $\frac{21}{2}$ , l1 -> 5, l2 -> 10,
 $\theta_1 \rightarrow \text{ArcTan}\left[\frac{47}{\sqrt{17391}}\right]$ ,  $\theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{247}{\sqrt{17391}}\right]$ },
{posSlider -> 10, l1 -> 5, l2 -> 10,  $\theta_1 \rightarrow \text{ArcTan}\left[\frac{1}{\sqrt{15}}\right]$ ,  $\theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{7}{\sqrt{15}}\right]$ },
{posSlider ->  $\frac{19}{2}$ , l1 -> 5, l2 -> 10,
 $\theta_1 \rightarrow \text{ArcTan}\left[\frac{61}{21\sqrt{319}}\right]$ ,  $\theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{661}{21\sqrt{319}}\right]$ },
{posSlider -> 9, l1 -> 5, l2 -> 10,  $\theta_1 \rightarrow \text{ArcTan}\left[\frac{1}{4\sqrt{14}}\right]$ ,  $\theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{13}{2\sqrt{14}}\right]$ },
{posSlider ->  $\frac{17}{2}$ , l1 -> 5, l2 -> 10,
```

$$\begin{aligned} & \theta_1 \rightarrow -\text{ArcTan}\left[\frac{11}{3\sqrt{12831}}\right], \theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{589}{3\sqrt{12831}}\right]}, \\ & \{\text{posSlider} \rightarrow 8, 11 \rightarrow 5, 12 \rightarrow 10, \theta_1 \rightarrow -\text{ArcTan}\left[\frac{11}{\sqrt{6279}}\right], \theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{139}{\sqrt{6279}}\right]\}, \\ & \{\text{posSlider} \rightarrow \frac{15}{2}, 11 \rightarrow 5, 12 \rightarrow 10, \theta_1 \rightarrow -\text{ArcTan}\left[\frac{1}{\sqrt{15}}\right], \theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{7}{\sqrt{15}}\right]\}, \\ & \{\text{posSlider} \rightarrow 7, 11 \rightarrow 5, 12 \rightarrow 10, \theta_1 \rightarrow -\text{ArcTan}\left[\frac{13}{4\sqrt{66}}\right], \theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{31}{2\sqrt{66}}\right]\}, \\ & \{\text{posSlider} \rightarrow \frac{13}{2}, 11 \rightarrow 5, 12 \rightarrow 10, \\ & \theta_1 \rightarrow -\text{ArcTan}\left[\frac{131}{\sqrt{50439}}\right], \theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{469}{\sqrt{50439}}\right]\}, \\ & \{\text{posSlider} \rightarrow 6, 11 \rightarrow 5, 12 \rightarrow 10, \theta_1 \rightarrow -\text{ArcTan}\left[\frac{13}{\sqrt{231}}\right], \theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{37}{\sqrt{231}}\right]\}, \\ & \{\text{posSlider} \rightarrow \frac{11}{2}, 11 \rightarrow 5, 12 \rightarrow 10, \theta_1 \rightarrow -\text{ArcTan}\left[\frac{179}{\sqrt{16359}}\right], \\ & \theta_2 \rightarrow -\pi + \text{ArcTan}\left[\frac{421}{\sqrt{16359}}\right]\}, \{\text{posSlider} \rightarrow 5, 11 \rightarrow 5, 12 \rightarrow 10, \theta_1 \rightarrow -\frac{\pi}{2}, \theta_2 \rightarrow -\frac{\pi}{2}\} \end{aligned}$$

Display the trajectory if the $\{\theta_1, \theta_2\}$ coordinate frame

```
In[39]:= gr1 = ListPlot[{Mod[θ1, 2 π], Mod[θ2, 2 π]} /. ls1,
  PlotJoined → True, PlotStyle → {Red, Thickness[0.01]]}
```



Out[39]= - Graphics -

Move the slider from position 5 to 15 by using the second solution branch, and calculate the substitution values of all parameters

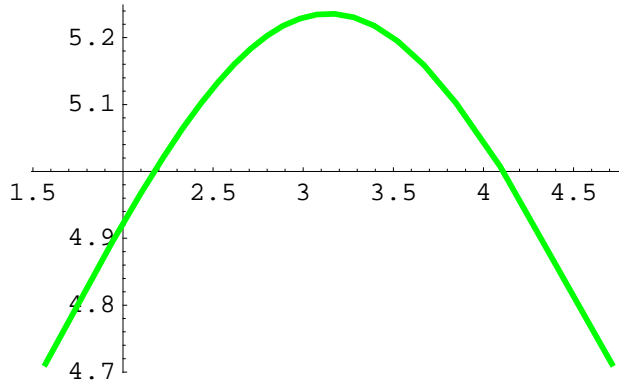
```
In[40]:= ls2 = GetLinkageRules[cr2, {{posSlider -> 5}, {posSlider -> 15}},
  Resolution → 20, SubstituteParameters -> True]
```

```
Out[40]= {{posSlider → 5, 11 → 5, 12 → 10, θ1 → -π/2, θ2 → -π/2}, {posSlider → 11/2,
  11 → 5, 12 → 10, θ1 → -π + ArcTan[179/√16359], θ2 → -ArcTan[421/√16359]}},
  {posSlider → 6, 11 → 5, 12 → 10, θ1 → -π + ArcTan[13/√231], θ2 → -ArcTan[37/√231]}},
```

$$\begin{aligned}
& \left\{ \text{posSlider} \rightarrow \frac{13}{2}, 11 \rightarrow 5, 12 \rightarrow 10, \right. \\
& \quad \left. \theta 1 \rightarrow -\pi + \text{ArcTan}\left[\frac{131}{\sqrt{50439}}\right], \theta 2 \rightarrow -\text{ArcTan}\left[\frac{469}{\sqrt{50439}}\right] \right\}, \\
& \left\{ \text{posSlider} \rightarrow 7, 11 \rightarrow 5, 12 \rightarrow 10, \theta 1 \rightarrow -\pi + \text{ArcTan}\left[\frac{13}{4\sqrt{66}}\right], \theta 2 \rightarrow -\text{ArcTan}\left[\frac{31}{2\sqrt{66}}\right] \right\}, \\
& \left\{ \text{posSlider} \rightarrow \frac{15}{2}, 11 \rightarrow 5, 12 \rightarrow 10, \theta 1 \rightarrow -\pi + \text{ArcTan}\left[\frac{1}{\sqrt{15}}\right], \theta 2 \rightarrow -\text{ArcTan}\left[\frac{7}{\sqrt{15}}\right] \right\}, \\
& \left\{ \text{posSlider} \rightarrow 8, 11 \rightarrow 5, 12 \rightarrow 10, \theta 1 \rightarrow -\pi + \text{ArcTan}\left[\frac{11}{\sqrt{6279}}\right], \theta 2 \rightarrow -\text{ArcTan}\left[\frac{139}{\sqrt{6279}}\right] \right\}, \\
& \left\{ \text{posSlider} \rightarrow \frac{17}{2}, 11 \rightarrow 5, 12 \rightarrow 10, \right. \\
& \quad \left. \theta 1 \rightarrow -\pi + \text{ArcTan}\left[\frac{11}{3\sqrt{12831}}\right], \theta 2 \rightarrow -\text{ArcTan}\left[\frac{589}{3\sqrt{12831}}\right] \right\}, \\
& \left\{ \text{posSlider} \rightarrow 9, 11 \rightarrow 5, 12 \rightarrow 10, \theta 1 \rightarrow \pi - \text{ArcTan}\left[\frac{1}{4\sqrt{14}}\right], \theta 2 \rightarrow -\text{ArcTan}\left[\frac{13}{2\sqrt{14}}\right] \right\}, \\
& \left\{ \text{posSlider} \rightarrow \frac{19}{2}, 11 \rightarrow 5, 12 \rightarrow 10, \right. \\
& \quad \left. \theta 1 \rightarrow \pi - \text{ArcTan}\left[\frac{61}{21\sqrt{319}}\right], \theta 2 \rightarrow -\text{ArcTan}\left[\frac{661}{21\sqrt{319}}\right] \right\}, \\
& \left\{ \text{posSlider} \rightarrow 10, 11 \rightarrow 5, 12 \rightarrow 10, \theta 1 \rightarrow \pi - \text{ArcTan}\left[\frac{1}{\sqrt{15}}\right], \theta 2 \rightarrow -\text{ArcTan}\left[\frac{7}{\sqrt{15}}\right] \right\}, \\
& \left\{ \text{posSlider} \rightarrow \frac{21}{2}, 11 \rightarrow 5, 12 \rightarrow 10, \right. \\
& \quad \left. \theta 1 \rightarrow \pi - \text{ArcTan}\left[\frac{47}{\sqrt{17391}}\right], \theta 2 \rightarrow -\text{ArcTan}\left[\frac{247}{\sqrt{17391}}\right] \right\}, \\
& \left\{ \text{posSlider} \rightarrow 11, 11 \rightarrow 5, 12 \rightarrow 10, \theta 1 \rightarrow \pi - \text{ArcTan}\left[\frac{23}{8\sqrt{39}}\right], \theta 2 \rightarrow -\text{ArcTan}\left[\frac{49}{4\sqrt{39}}\right] \right\}, \\
& \left\{ \text{posSlider} \rightarrow \frac{23}{2}, 11 \rightarrow 5, 12 \rightarrow 10, \right. \\
& \quad \left. \theta 1 \rightarrow \pi - \text{ArcTan}\left[\frac{229}{\sqrt{159159}}\right], \theta 2 \rightarrow -\text{ArcTan}\left[\frac{829}{\sqrt{159159}}\right] \right\}, \\
& \left\{ \text{posSlider} \rightarrow 12, 11 \rightarrow 5, 12 \rightarrow 10, \theta 1 \rightarrow \pi - \text{ArcTan}\left[\frac{23}{3\sqrt{119}}\right], \theta 2 \rightarrow -\text{ArcTan}\left[\frac{73}{3\sqrt{119}}\right] \right\}, \\
& \left\{ \text{posSlider} \rightarrow \frac{25}{2}, 11 \rightarrow 5, 12 \rightarrow 10, \theta 1 \rightarrow \pi - \text{ArcTan}\left[\frac{13}{\sqrt{231}}\right], \theta 2 \rightarrow -\text{ArcTan}\left[\frac{37}{\sqrt{231}}\right] \right\}, \\
& \left\{ \text{posSlider} \rightarrow 13, 11 \rightarrow 5, 12 \rightarrow 10, \theta 1 \rightarrow \pi - \text{ArcTan}\left[\frac{47}{12\sqrt{14}}\right], \theta 2 \rightarrow -\text{ArcTan}\left[\frac{61}{6\sqrt{14}}\right] \right\}, \\
& \left\{ \text{posSlider} \rightarrow \frac{27}{2}, 11 \rightarrow 5, 12 \rightarrow 10, \right. \\
& \quad \left. \theta 1 \rightarrow \pi - \text{ArcTan}\left[\frac{143}{\sqrt{11951}}\right], \theta 2 \rightarrow -\text{ArcTan}\left[\frac{343}{\sqrt{11951}}\right] \right\}, \\
& \left\{ \text{posSlider} \rightarrow 14, 11 \rightarrow 5, 12 \rightarrow 10, \theta 1 \rightarrow \pi - \text{ArcTan}\left[\frac{121}{3\sqrt{551}}\right], \theta 2 \rightarrow -\text{ArcTan}\left[\frac{271}{3\sqrt{551}}\right] \right\}, \\
& \left\{ \text{posSlider} \rightarrow \frac{29}{2}, 11 \rightarrow 5, 12 \rightarrow 10, \theta 1 \rightarrow \pi - \text{ArcTan}\left[\frac{541}{\sqrt{43719}}\right], \right. \\
& \quad \left. \theta 2 \rightarrow -\text{ArcTan}\left[\frac{1141}{\sqrt{43719}}\right] \right\}, \left\{ \text{posSlider} \rightarrow 15, 11 \rightarrow 5, 12 \rightarrow 10, \theta 1 \rightarrow \frac{\pi}{2}, \theta 2 \rightarrow -\frac{\pi}{2} \right\}
\end{aligned}$$

Display the trajectory if the $\{\theta_1, \theta_2\}$ coordinate frame

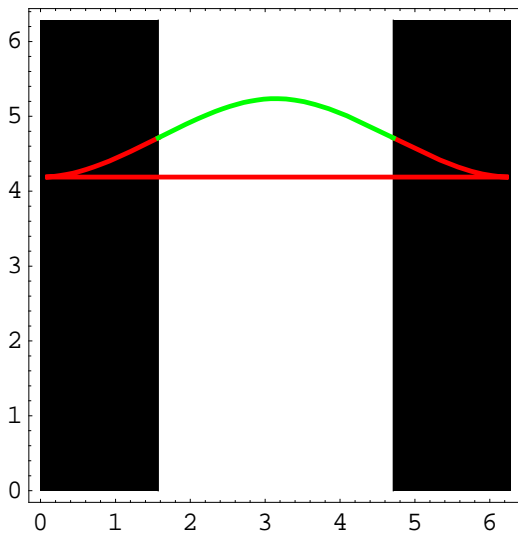
```
In[41]:= gr2 = ListPlot[{Mod[ $\theta_1$ ,  $2\pi$ ], Mod[ $\theta_2$ ,  $2\pi$ ]} /. ls2,
  PlotJoined → True, PlotStyle → {Green, Thickness[0.01]]}
```



Out[41]= - Graphics -

Display the trajectory if the $\{\theta_1, \theta_2\}$ coordinate frame as the slider moves from position 15 to 5 and then back to position 5

```
In[42]:= Show[{gr, gr1, gr2}]
```



Out[42]= - Graphics -

Animate the linkage to verify the result

```
In[48]:= ls = AnimateLinkage[cr, Join[Drop[ls1, 1], Drop[ls2, 1]], Resolution → None,
  LinkMarkers → All, PlotRange → {{-10, 10}, {-30, 10}, {-1, 5}},
  ViewPoint → {-1, 0, 100}, MarkerSize → 4, Axes → True, FaceGrids → {{0, 0, 1}}];
```